

# On the Nonlocality in the Coulomb Gauge External Field Problem

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## Abstract

The apparent nonlocality of the Coulomb gauge external field problem in electrodynamics is illustrated with an example in which nonlocality is especially striking. A rather intuitive derivation of this apparent nonlocal behaviour from a purely local picture is given. In the course of the derivation Lorentz-force naturally decomposes into a field and source part and a novel classical effect based on this separation is pointed out.

## 1 Az introductory example

Consider a rigid body with a given charge distribution  $\rho(\mathbf{x})$  on it but of total charge equal to zero. If it is placed into a homogeneous magnetic field  $\mathbf{B} = \nabla \times \mathbf{A}$  then in the combined electric field  $\mathbf{E}$  of the body and the external magnetic field a certain amount  $\mathbf{N}$  of angular momentum will arise. The simplest way to demonstrate this is to slowly remove the magnetic field and to calculate the torque

$$\mathbf{K}(t) = \int d^3x [\mathbf{x}, \rho \mathbf{E}'(t)] \quad (1)$$

acting on the body due to the electric field  $\mathbf{E}'(t)$  induced by the changing  $\mathbf{B}(\mathbf{t})$ . The total amount of angular momentum imparted to the body is equal to the angular momentum  $\mathbf{N}$ .

In Coulomb gauge the vector potential is chosen so as to make its divergence to vanish and the scalar potential  $\phi$  is equal to the Coulomb field of the charge density  $\rho(\mathbf{x})$ . For a homogeneous field we have<sup>1</sup>

$$\mathbf{A} = \frac{1}{2}[\mathbf{B}, \mathbf{x}] \quad (\nabla \cdot \mathbf{A} = 0). \quad (2)$$

The induced electric field is determined by the equation

$$\mathbf{E}'(t) = -\frac{1}{c}\dot{\mathbf{A}}(t), \quad \mathbf{A}(0) = \mathbf{A}, \quad \mathbf{A}(\infty) = 0.$$

Then (1) can be written as

$$\mathbf{K}(t) = -\frac{d}{dt} \int d^3x [\mathbf{x}, \frac{\rho}{c}\mathbf{A}(t)].$$

Since  $\dot{\mathbf{N}} = \mathbf{K}$  the total amount of angular momentum received by the body is equal to

$$\mathbf{N} = + \int d^3x [\mathbf{x}, \frac{\rho}{c}\mathbf{A}]. \quad (3)$$

The angular momentum stored in the field can, therefore, be written as

$$\mathbf{N} = \frac{1}{2c} \int d^3x \rho(\mathbf{x}) [\mathbf{x}, [\mathbf{B}, \mathbf{x}]].$$

Since

$$[\mathbf{x}, [\mathbf{B}, \mathbf{x}]] = -\frac{1}{3}(3(\mathbf{x} \cdot \mathbf{B})\mathbf{x} - r^2\mathbf{B}) + \frac{2}{3}r^2\mathbf{B},$$

where  $r^2 \equiv \mathbf{x}^2$ , the Cartesian components of  $\mathbf{N}$  are

$$N_\alpha = -\frac{1}{6c}D_{\alpha\beta}B_\beta + \frac{1}{3c}\langle \rho r^2 \rangle B_\alpha. \quad (4)$$

Here

$$D_{\alpha\beta} = \int d^3x \rho(\mathbf{x})(3x_\alpha x_\beta - r^2\delta_{\alpha\beta})$$

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<sup>1</sup>The condition  $\nabla \cdot \mathbf{A} = 0$  would permit us to add to  $\mathbf{A}$  a gradient of a harmonic function but for the calculation of the gauge invariant quantity (1) the choice of the gauge is dictated solely by convenience. We note in passing that in this static situation there is in fact no difference between Coulomb and Lorentz gauges.

is the quadrupole momentum tensor of the body and

$$\langle \rho r^2 \rangle = \int d^3x \rho(\mathbf{x}) r^2$$

is its scalar part.

Simple as it seems our example reveals its rather peculiar feature when one tries to recalculate  $\mathbf{N}$  starting from the equation

$$\mathbf{N} = \frac{1}{4\pi c} \int d^3x [\mathbf{x}, [\mathbf{E}, \mathbf{B}]] \quad (5)$$

which defines field angular momentum in terms of field strengths. To this end we replace the fields in (5) by their respective potentials  $\mathbf{B} = \nabla \times \mathbf{A}$  and  $\mathbf{E} = -\nabla\phi$  and perform partial integrations within a sphere of arbitrary large radius  $R$ . Though, neglecting surface terms, we arrive at the expression (4) again, the contribution of the surface terms turns out to be of finite magnitude independent of  $R$  (see Appendix):

$$\text{Surface Terms} = \frac{1}{10c} D_{\alpha\beta} B_\beta. \quad (6)$$

Hence we obtain that, contrary to (4), within any sphere surrounding the body the amount of angular momentum is equal to

$$N_\alpha^{(R)} = -\frac{1}{15c} D_{\alpha\beta} B_\beta + \frac{1}{3c} \langle \rho r^2 \rangle B_\alpha. \quad (7)$$

Moreover, this momentum is concentrated within the volume of the body since, owing to the independence of (6) of  $R$ , in any spherical shell outside the body the value of the angular momentum is cancelled to zero.

The conclusion from this apparent contradiction is that the notion of the homogeneous magnetic field of infinite extension is in general an unacceptable idealization. In local problems as e.g. in Zeeman effect, or even in the derivation of eq. (4), the concept of a magnetic field of infinite extension is a perfectly suitable idealization but when total field momentum or angular momentum are to be calculated the sources of  $\mathbf{B}$  should be taken into consideration explicitly.

Their simple realization is an infinitely long ideal straight solenoid (an idealization too but this time a harmless one) of very large circular cross section of radius  $R_s$  with our body situated on its axis. Then an amount

of angular momentum given by (7) is concentrated within the volume of the body while the remaining part of it, the negative of (6), is found within the solenoid above and below the sphere of radius  $R_s$  (see Appendix). The sum total is then given by (4) as expected.

## 2 The problem of apparent nonlocality

Assume now that the body we are considering is capable to rotate around a fixed point of it. Then its Lagrangian is

$$L = \frac{1}{2} I_{\alpha\beta} \Omega_\alpha \Omega_\beta + \frac{1}{c} \int d^3x \rho(\mathbf{x}) \mathbf{A}(\mathbf{x}) \cdot \mathbf{V}(\mathbf{x}),$$

in which  $I_{\alpha\beta}$  and  $\boldsymbol{\Omega}$  are the inertia tensor and the angular velocity and  $\mathbf{V} = [\boldsymbol{\Omega}, \mathbf{x}]$ . Using (3) and the relation  $\mathbf{A}[\boldsymbol{\Omega}, \mathbf{x}] = \boldsymbol{\Omega}[\mathbf{x}, \mathbf{A}]$ ,  $L$  can be written as

$$L = \frac{1}{2} I_{\alpha\beta} \Omega_\alpha \Omega_\beta + \mathbf{N} \cdot \boldsymbol{\Omega}.$$

In the Hamiltonian  $H = \mathbf{J} \cdot \boldsymbol{\Omega} - L$

$$J_\alpha = \frac{\partial L}{\partial \Omega_\alpha} = I_{\alpha\beta} \Omega_\beta + N_\alpha.$$

From this

$$\Omega_\alpha = (I^{-1})_{\alpha\beta} (J_\beta - N_\beta).$$

Substituting this into  $H$  we obtain after some rearrangements

$$H = \frac{1}{2} (I^{-1})_{\alpha\beta} (J_\alpha - N_\alpha) \cdot (J_\beta - N_\beta). \quad (8)$$

The canonical variables in this Hamiltonian are the Euler angles which determine the orientation of our rotator and their conjugate momenta. We do not recall the explicit dependence of  $\mathbf{J}$  and  $\Omega_{\alpha\beta}$  on these variables. According to (4)  $\mathbf{N}$  depends on the Euler angles through the dipole momentum tensor  $D_{\alpha\beta}$ .

The Hamiltonian is the energy of the rotator. The Lorentz-force by which  $\mathbf{B}$  acts on the rotator does not perform any work on it and so  $H$  is equal to its kinetic energy. Hence, the difference  $(\mathbf{J} - \mathbf{N})$  must be equal to the angular momentum  $\mathbf{R}$  of its rotation. The canonical angular momentum  $\mathbf{J}$  can,

therefore, be interpreted as the sum of the rotational angular momentum and of the angular momentum stored in the combined electric field of the body and external magnetic field.

In the previous section we have found that, depending on  $D_{\alpha\beta}$  (i.e. on the orientation of the body), a well defined fraction of the field angular momentum is found at a distance arbitrarily far (say light years away) from the body at rest. Remember now that the most characteristic feature of the Coulomb gauge is that the potential  $\phi$  of a time dependent charge distribution  $\rho(\mathbf{x}, t)$  is equal to its *instantaneous* Coulomb potential [1]. Hence, when our body is allowed to rotate then the potential  $\phi$  determining field angular momentum rotates together with it. Hence, in spite of the huge distance, the amount of this quantity at the far ends of the solenoid varies synchronously with the rotation of the body without any delay. Though from a purely computational point of view this apparent nonlocal behaviour presents no difficulty a simple explanation of it in terms of local interactions alone would certainly improve our understanding of this important aspect of electrodynamics<sup>2</sup>. For this reason in the next section the results arrived at above will be rederived by means of explicitly local considerations.

### 3 The underlying physical picture

Consider now a point charge moving arbitrarily along the trajectory  $\mathbf{x} = \mathbf{r}(t)$  in a stationary magnetic field  $\mathbf{B}(\mathbf{x})$  of *finite extension*. The Hamiltonian is

$$H = \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}) \right)^2.$$

Since the Lorentz-force acting on the charge does not perform any work on it this  $H$  is numerically equal to the kinetic energy  $mV^2/2$ . Hence  $\mathbf{p} = m\mathbf{V} + (e/c)\mathbf{A}(\mathbf{r})$  where  $\mathbf{V} = \dot{\mathbf{r}}(t)$  is the instantaneous velocity of the charge.

Any gauge transformation of  $\mathbf{A}$  is accompanied by a canonical transformation of  $\mathbf{p}$  so as to make the difference  $\mathbf{p} - (e/c)\mathbf{A}$  unchanged. As it is well

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<sup>2</sup>Another problem related to the nonlocal behaviour of the Coulomb gauge potential is the finiteness of signal (light) velocity. A substantial amount of literature has been devoted to this subject since W. Heisenberg proposed it to his disciple S. Kikuchi [2] and E. Fermi published his study of quantum electrodynamics [3]. A recent discussion of this problem in a variety of gauges is found in [4] where references to earlier literature are also found.

known (see e.g. [5]) in the special case of the Coulomb-gauge the quantity  $(e/c)\mathbf{A}$  is equal to the momentum  $\mathbf{G}$  stored in the combined electric field of the charge and the external field  $\mathbf{B}$  when the charge is *at rest* at the point  $\mathbf{r}(t)$  of its trajectory. The field momentum, therefore, seems as rigidly attached to the particle as was the case with the field angular momentum  $\mathbf{N}$  and the rotator in the example of the previous sections. At any moment of time  $t$  and point  $P$  in space the electric field contributing to  $\mathbf{G}$  is determined by the location of the charge  $\mathbf{r}(t)$  at the same moment of time however far away from  $P$ . Since composite charged bodies are built up of pointlike charged constituents the nonlocality found in the case of the rotating body has been inherited from this description of the point charge.

The sole virtue of the example with the rotator in the previous section was that, owing to the splitting of  $\mathbf{N}$  into fractions located within two disjoint domains of space, its nonlocality appeared more spectacular than that of a point charge without structure. But in order to elucidate the physical picture behind this seemingly nonlocal Hamiltonian description it is sufficient to confine ourselves to the case of a single point charge.

The electromagnetic part of the energy-momentum tensor decomposes into the sum of three terms  $T_{\alpha\beta}^{(i)}$  according to the power of  $\mathbf{B}$  in them. Each term satisfies a balance equation (continuity equation with sources) of its own. This may be verified either by direct computation or by rescaling the external field  $\mathbf{B}$  with a scalar factor  $k$ . For an arbitrary value of  $k$  the electromagnetic energy-momentum tensor becomes equal to the sum  $k^0 T_{\alpha\beta}^{(0)} + k^1 T_{\alpha\beta}^{(1)} + k^2 T_{\alpha\beta}^{(2)}$ . Since the sum satisfies a balance equation for any value of  $k$  the three parts must satisfy it separately.

The term of second order, being constant in time, is obviously of no interest. The term independent of the external field describes radiation and radiation reaction (self-interaction) of the charge and obeys a balance equation with a source equal to the negative of the self-force. This term is of paramount importance of its own right but for the external field problems only the mixed part  $T_{\alpha\beta}^{(1)}$  of the tensor is of significance. The sources of the balance equation satisfied by this term in an external magnetic field are the negative of the Lorentz-force acting on the charge in the external field and the minus Lorentz-force density experienced by the sources of this field from the side of the moving charge. From our point of view a cardinal property of this part of the energy-momentum tensor is the absence of radiation. This follows from the finite extension of the external field in space.

The force experienced by the charge at moment  $t_0$  does not depend on its subsequent motion. Therefore, in calculating this force we can rely on the *truncated* (at  $t = t_0$ ) *trajectory* which is obtained from the true trajectory by bringing the charge to a stop at the position  $\mathbf{r}_0 = \mathbf{r}(t_0)$ . At this moment the true spatial distribution of the momentum density is certainly different from the momentum density  $(1/4\pi c)[\mathbf{E}^{(0)}, \mathbf{B}]$  of a charge in state of rest at  $\mathbf{r}_0$  but the total field momentum will tend to the integral

$$\mathbf{G}^{(0)} = \frac{1}{4\pi c} \int d^3x [\mathbf{E}^{(0)}, \mathbf{B}] \quad (9)$$

as  $t \rightarrow \infty$ <sup>3</sup>. The reason is that due to the absence of radiation the mixed field momentum emitted by the moving charge before  $t_0$  will be completely absorbed in later times by the sources of the external field.

Let us compare momentum exchange between the particle and the field in the two motions truncated at moments of time  $t_0$  and  $t_0 + dt_0$ . The field momentum emitted by the particle in the infinitesimal interval  $(t_0 < t < t_0 + dt_0)$  consists of two parts: that what remains stored in the form of field momentum forever and an additional part which will be subsequently absorbed by the sources of the external field. Accordingly, the force experienced by the particle consists of two contributions:

$$\mathbf{F}^{(0)} = -\frac{d\mathbf{G}^{(0)}}{dt_0} - \mathbf{F}'^{(0)}, \quad (10)$$

where  $\mathbf{F}'^{(0)}$  is the Lorentz-force experienced by the sources of  $\mathbf{B}$  when some portion of the mixed field momentum emitted at  $t_0$  by the charge is absorbed.

Since  $\mathbf{G}^{(0)}$  depends on time through the motion of the charge we have

$$\frac{dG_\alpha^{(0)}}{dt_0} = V_\beta(t_0) \partial_\beta G_\alpha^{(0)}, \quad (11)$$

where  $\partial$  denotes derivation with respect to  $\mathbf{r}$ .

The force  $\mathbf{F}'^{(0)}$  is related to the momentum difference received by the sources of  $\mathbf{B}$  in the motions truncated at the moments  $t_0 + dt_0$  and  $t_0$ . Hence, in order to evaluate this force we must first calculate the momentum  $\mathbf{G}'^{(0)}$  received by the sources via the retarded field of the charge. It is equal to the time integral of the Lorentz-force experienced by them:

$$\mathbf{G}'^{(0)} = \frac{1}{c} \int dt \cdot d^3x [\mathbf{J}(\mathbf{x}), \mathbf{B}'^{(0)}(\mathbf{x}, t)], \quad (12)$$

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<sup>3</sup>The upper index zero on a quantity indicates that it refers to the truncated motion.

where  $\mathbf{J}(\mathbf{x})$  is the current density supporting the external magnetic field. The retarded magnetic field of the charged particle is

$$\mathbf{B}'^{(0)}(\mathbf{x}, t) = \nabla \times \mathbf{A}'^{(0)}(\mathbf{x}, t).$$

The vector potential here is the Liénard-Wiechert potential of the particle's current density  $\mathbf{j}^{(0)}$ :

$$\mathbf{A}'^{(0)}(\mathbf{x}, t) = \frac{1}{c} \int dt' \cdot d^3x' \frac{\delta(t' - t + |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} \mathbf{j}^{(0)}(\mathbf{x}', t'). \quad (13)$$

Since at  $t = t_0$  the particle comes to a stop we can write

$$\mathbf{j}^{(0)}(\mathbf{x}', t') = \mathbf{j}(\mathbf{x}', t') \Theta(t_0 - t'),$$

where  $\Theta(t)$  is the step-function. Then

$$\int_{-\infty}^{\infty} dt \mathbf{A}'^{(0)}(\mathbf{x}, t) = \frac{1}{c} \int \frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} \int_{-\infty}^{t_0} dt' \mathbf{j}(\mathbf{x}', t').$$

Now, using the relation

$$\nabla \times (f\mathbf{U}) = [\nabla f, \mathbf{U}] + f\nabla \times \mathbf{U} \quad (14)$$

we calculate the curl of this function:

$$\begin{aligned} \int_{-\infty}^{\infty} dt \mathbf{B}'^{(0)}(\mathbf{x}, t) &= \nabla \times \int_{-\infty}^{\infty} dt \mathbf{A}'^{(0)}(\mathbf{x}, t) = \\ &= \frac{1}{c} \int d^3x' \left[ \nabla |\mathbf{x} - \mathbf{x}'|^{-1}, \int_{-\infty}^{t_0} dt' \mathbf{j}(\mathbf{x}', t') \right]. \end{aligned}$$

To obtain  $\mathbf{F}'^{(0)}$  we have to compute the derivative of  $\mathbf{G}'^{(0)}$  as given in (12) with respect to  $t_0$ . At first we write

$$\frac{d}{dt_0} \int_{-\infty}^{\infty} dt \mathbf{B}'^{(0)}(\mathbf{x}, t) = \frac{1}{c} \int d^3x' [\nabla |\mathbf{x} - \mathbf{x}'|^{-1}, \mathbf{j}(\mathbf{x}', t_0)].$$

The current density of a pointlike charged particle is

$$\mathbf{j}(\mathbf{x}', t_0) = e\mathbf{V}(t_0)\delta(\mathbf{x}' - \mathbf{r}(t_0)).$$

Then

$$\frac{d}{dt_0} \int_{-\infty}^{\infty} dt \mathbf{B}'^{(0)}(\mathbf{x}, t) = \frac{e}{c} [\nabla |\mathbf{x} - \mathbf{r}(t_0)|^{-1}, \mathbf{V}(t_0)] = \frac{1}{c} [\nabla \phi^{(0)}, \mathbf{V}(t_0)], \quad (15)$$

where

$$\phi^{(0)}(\mathbf{x}) = \frac{e}{|\mathbf{x} - \mathbf{r}(t_0)|}$$

is the potential of a charge at  $\mathbf{x} = \mathbf{r}(t_0)$ .

Now we can write

$$\begin{aligned} \mathbf{F}'^{(0)} &= \frac{d\mathbf{G}'^{(0)}}{dt_0} = \frac{1}{c^2} \int d^3x [\mathbf{J}(\mathbf{x}), [\nabla \phi^{(0)}, \mathbf{V}(t_0)]] = \\ &= \frac{1}{c^2} \int d^3x \{ (\mathbf{J} \cdot \mathbf{V}(t_0)) \nabla \phi^{(0)} - (\mathbf{J} \cdot \nabla \phi^{(0)}) \mathbf{V}(t_0) \} \end{aligned}$$

The second integral on the r.h.s. is equal to zero because the stationary current  $\mathbf{J}(\mathbf{x})$  is solenoidal:

$$\int d^3x \mathbf{J} \cdot \nabla \phi^{(0)} = - \int d^3x \phi^{(0)} \nabla \cdot \mathbf{J} = 0.$$

In Cartesian coordinates we have

$$\nabla_{\alpha} \phi^{(0)} \equiv \frac{\partial \phi^{(0)}}{\partial x_{\alpha}} = - \frac{\partial \phi^{(0)}}{\partial r_{\alpha}} \equiv -\partial_{\alpha} \phi^{(0)}.$$

Then

$$\mathbf{F}'^{(0)} = -\frac{1}{c^2} \partial_{\alpha} \int d^3x (\mathbf{J} \cdot \mathbf{V}(t_0)) \phi^{(0)} = -V_{\beta}(t_0) \partial_{\alpha} \left[ \frac{1}{c^2} \int d^3x \phi^{(0)} J_{\beta} \right]. \quad (16)$$

Equation  $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}$  permits us to replace  $\mathbf{J}$  by  $\mathbf{B}$ :

$$\frac{1}{c^2} \int d^3x \phi^{(0)} J_{\beta} = \frac{1}{4\pi c} \int d^3x \phi^{(0)} \nabla \times \mathbf{B}.$$

Here we can use (14) again:

$$\frac{1}{c^2} \int d^3x \phi^{(0)} J_{\beta} = \frac{1}{4\pi c} \left\{ \int d^3x [\mathbf{B}, \nabla \phi^{(0)}] + \int d^3x \nabla \times (\phi^{(0)} \mathbf{B}) \right\}$$

The second term is transformed into a surface integral and can be dropped. In the first term  $\nabla\phi^{(0)} = -\mathbf{E}^{(0)}$ . Then, using (9), we obtain

$$\frac{1}{c^2} \int d^3x \phi^{(0)} J_\beta = \frac{1}{4\pi c} \int d^3x [\mathbf{E}^{(0)}, \mathbf{B}] = \mathbf{G}^{(0)}.$$

Substitution of this expression into (16) leads to the concise form

$$F'_\alpha{}^{(0)} = -V_\beta(t_0) \partial_\alpha G_\beta^{(0)}. \quad (17)$$

From now on we may drop upper 0 indices. Inserting (17) and (11) into (10) we obtain for the force acting on the charge the formula

$$F_\alpha = V_\beta \partial_\alpha G_\beta - V_\beta \partial_\beta G_\alpha, \quad (18)$$

where  $\mathbf{V}(t) = \dot{\mathbf{r}}(t)$  and  $\mathbf{G}$  is equal to the field momentum stored in the combined field of a charge *resting at*  $\mathbf{r}(t)$ . The vector form of this equation is

$$\mathbf{F} = [\mathbf{V}, \nabla \times \mathbf{G}],$$

where  $\nabla$  denotes now differentiation with respect to  $\mathbf{r}$  the only coordinate  $\mathbf{G}$  depends on.

On the other hand, we know that  $\mathbf{F}$  is the Lorentz-force which acts on the charge:

$$\mathbf{F} = \frac{e}{c} [\mathbf{V}, \mathbf{B}] = \left[ \mathbf{V}, \nabla \times \frac{e}{c} \mathbf{A} \right].$$

We see that (up to the factor  $e/c$ )  $\mathbf{G}$  can be identified with the Coulomb gauge vector potential provided the latter can be chosen vanishing together with  $\mathbf{G}$  and this choice is unique (see the example at the end of this section). Equation (18) can, therefore, be written also as<sup>4</sup>

$$F_\alpha = \frac{e}{c} (V_\beta \partial_\alpha A_\beta - V_\beta \partial_\beta A_\alpha). \quad (19)$$

At the beginning of the present section we have stressed the apparent nonlocality of this gauge. Now we see explicitly that locality is by no means

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<sup>4</sup>Since  $\mathbf{V}$  is independent of  $\mathbf{r}$  the covariant form of this equation is

$$\mathbf{F} = \frac{e}{c} (\nabla(\mathbf{V} \cdot \mathbf{A}) - (\mathbf{V} \cdot \nabla) \mathbf{A}).$$

As explained in the text in Coulomb gauge both terms on the r. h. s. have their own physical meaning separately.

broken: There is in fact no field momentum attached rigidly to the charge and in the general case the true mixed field momentum differs from  $\mathbf{G}$  at any moment of time. The flow of the combined field momentum density obeys continuity equation in space between the moving charge and the sources of the external field and this is what locality means. Moreover, the physical picture revealed provides the interpretation of the terms in the formula (18) for the force  $\mathbf{F}$  experienced by the moving charge. They describe momentum exchange of two different types: the first one with the sources of the external field and the second one with the field itself.

An interesting special example is that of a charged particle passing by an infinitely long straight ideal solenoid (a classical ‘Aharonov-Bohm situation’). Since the magnetic field outside the solenoid is zero the two terms in (18) cancel each other but neither of them vanishes. That means that the charge ‘catalyzes’ momentum transfer between the combined field and the coil. Outside the solenoid we have

$$\mathbf{A} = \frac{S}{2\pi} \cdot \frac{[\mathbf{B}, \mathbf{r}]}{r^2}, \quad (20)$$

where  $S$  is the cross section of the solenoid. In this domain the two terms in (18) are equal to each other and so (17) can be written as  $F'_\alpha = -V_\beta \partial_\beta G_\alpha$ . Since  $\mathbf{G} = (e/c)\mathbf{A}$ , the total momentum imparted to the coil by the moment of time  $t$  the charge reaches position  $\mathbf{r}$  is equal to

$$\int_{-\infty}^t dt \mathbf{F}' = -\frac{eS}{2\pi c} \cdot \frac{[\mathbf{B}, \mathbf{r}]}{r^2}. \quad (21)$$

When the charge leaves freely to infinity no net momentum exchange occurs. But if it is stopped at  $\mathbf{r}$  a finite amount (21) of momentum is transferred from the combined field to the coil. I wonder whether this effect may be experimentally observed.

Inside the solenoid the continuation of the external vector potential (20) is given by (2) and, as it is well known, the field there is homogeneous. Inserting this into (19) we find that, for a point charge revolving in a plan perpendicular to solenoid’s symmetry axis on a circle around it, the two terms on the r. h. s. are equal to each other. Therefore, the reaction to the centripetal force acting on the charge is distributed evenly between the field and the coil: the force experienced by the coil is the half of the force experienced by the revolving point charge.

## Appendix. Derivation of eq. (7)

Let us denote the volume integral in (5) confined to a sphere of radius  $R$  by  $\mathbf{N}^{(R)}$ . Inserting  $\mathbf{E} = -\nabla\phi$  and  $\mathbf{B} = \nabla \times \mathbf{A}$  into it we obtain after some rearrangements

$$N_{\alpha}^{(R)} = -\frac{1}{4\pi c} \epsilon_{\alpha\beta\gamma} \int_R d^3x \, x_{\beta} \{ \nabla_{\delta}\phi \cdot \nabla_{\gamma}A_{\delta} - \nabla_{\delta}\phi \cdot \nabla_{\delta}A_{\gamma} \},$$

where  $\epsilon_{\alpha\beta\gamma}$  is the Levi-Civita symbol.

In the second term the two  $\nabla$  symbols have identical indices. By partial integration we make them to merge into  $\nabla^2$ :

$$N_{\alpha}^{(R)} = -\frac{1}{4\pi c} \epsilon_{\alpha\beta\gamma} \int_R d^3x \, \{ x_{\beta} \nabla_{\delta}\phi \cdot \nabla_{\gamma}A_{\delta} - \nabla_{\delta}(x_{\beta} \cdot \nabla_{\delta}\phi \cdot A_{\gamma}) + \nabla_{\beta}\phi \cdot A_{\gamma} + x_{\beta} \cdot \nabla^2\phi \cdot A_{\gamma} \}.$$

In the first term the index of the  $\nabla$ -symbol and that of the vector potential are identical. By further partial integration we form  $\nabla \cdot \mathbf{A}$  out of them:

$$\begin{aligned} N_{\alpha}^{(R)} = & -\frac{1}{4\pi c} \epsilon_{\alpha\beta\gamma} \int_R d^3x \, \{ \nabla_{\delta}(x_{\beta} \cdot \phi \cdot \nabla_{\gamma}A_{\delta}) - \nabla_{\gamma}(\phi A_{\beta}) + \nabla_{\gamma}\phi \cdot A_{\beta} - \\ & - \phi \cdot x_{\beta} \nabla_{\gamma}(\nabla \cdot \mathbf{A}) - \nabla_{\delta}(x_{\beta} \cdot \nabla_{\delta}\phi \cdot A_{\gamma}) + \nabla_{\beta}\phi \cdot A_{\gamma} + x_{\beta} \cdot \nabla^2\phi \cdot A_{\gamma} \} \end{aligned}$$

By eq. (2) we have  $\nabla \cdot \mathbf{A} = 0$ . Out of the six terms left we have three volume terms but two of them cancel for symmetry reasons:

$$\epsilon_{\alpha\beta\gamma}(\nabla_{\gamma}\phi \cdot A_{\beta} + \nabla_{\beta}\phi \cdot A_{\gamma}) = 0.$$

If we insert  $\nabla^2\phi = -4\pi\rho$  into the remaining one we obtain the r.h.s. of (3) the integration being confined to the sphere of radius  $R$ :

$$\mathbf{N}^{(0)} = + \int_R d^3x \, \left[ \mathbf{x}, \frac{\rho}{c} \mathbf{A} \right].$$

According to (4), its value is equal to

$$N_{\alpha}^{(0)} = -\frac{1}{6c} D_{\alpha\beta} B_{\beta} + \frac{1}{3c} \langle \rho r^2 \rangle B_{\alpha}$$

for any sphere outside the body.

The three surface integrals are

$$\begin{aligned} N_\alpha^{(1)} &= +\frac{1}{4\pi c}\epsilon_{\alpha\beta\gamma}\int_R d^3x \nabla_\gamma(\phi A_\beta) \\ N_\alpha^{(2)} &= -\frac{1}{4\pi c}\epsilon_{\alpha\beta\gamma}\int_R d^3x \nabla_\delta(x_\beta\phi \cdot \nabla_\gamma A_\delta) \\ N_\alpha^{(3)} &= +\frac{1}{4\pi c}\epsilon_{\alpha\beta\gamma}\int_R d^3x \nabla_\delta(x_\beta \cdot \nabla_\delta\phi \cdot A_\gamma). \end{aligned}$$

All of them must be of the form  $(\eta_i/c)D_{\alpha\beta}B_\beta$  ( $i = 1, 2, 3$ ) and our task is to calculate the coefficients  $\eta_i$ .

Using Gauss theorem

$$\int_R d^3x \nabla_\alpha f = \oint_R dS_\alpha \cdot f = \oint R^2 d\Omega \frac{x_\alpha}{R} f,$$

where  $\Omega$  denotes solid angle, we have

$$N_\alpha^{(1)} = +\frac{1}{c}\epsilon_{\alpha\beta\gamma}\frac{R}{4\pi}\oint d\Omega x_\gamma\phi A_\beta. \quad (\text{A.1})$$

The body is uncharged by hypotheses and so no Coulomb potential contributes to  $\phi$ . The contribution of its dipole moment  $\mathbf{d}$  to  $\mathbf{N}^{(1)}$  is of the form *constant*  $\times [\mathbf{d}, \mathbf{B}]$  but the constant must be zero because it is a polar vector while  $\mathbf{N}^{(1)}$  is an axial one. From moments higher than quadrupole no vector can be formed with  $\mathbf{B}$ . Hence it is only the quadrupole potential

$$\phi = \frac{1}{2R^5}x_\alpha x_\beta D_{\alpha\beta} \quad (\text{A.2})$$

which contributes to the surface integrals.

For the vector potential we can substitute

$$A_\beta = -\frac{1}{2}\epsilon_{\beta\rho\sigma}x_\rho B_\sigma,$$

and we obtain

$$N_\alpha^{(1)} = -\frac{1}{2c}\epsilon_{\alpha\beta\gamma}\epsilon_{\beta\rho\sigma}B_\sigma\frac{R}{4\pi}\int d\Omega x_\gamma x_\rho\phi = -\frac{R}{8\pi c}\left\{B_\alpha R^2\int d\Omega \phi + B_\gamma\int d\Omega x_\alpha x_\gamma\phi\right\}.$$

The first integral vanishes because of the zero net charge condition. Substituting (A.2) into the second term and using the relation

$$\int d\Omega x_\gamma x_\mu x_\nu x_\alpha = \frac{4\pi R^4}{15}(\delta_{\gamma\mu}\delta_{\nu\alpha} + \delta_{\gamma\nu}\delta_{\mu\alpha} + \delta_{\gamma\alpha}\delta_{\mu\nu})$$

we arrive at the value  $\eta_1 = 1/30$ .

A completely analogous derivation leads to  $\eta_2 = -1/30$ .

The remaining surface integral is

$$N_\alpha^{(3)} = +\frac{1}{c}\epsilon_{\alpha\beta\gamma}\frac{R}{4\pi}\int d\Omega x_\beta x_\delta \nabla_\delta \phi \cdot A_\gamma.$$

Since the potential (A.2) is of degree  $-3$  in  $r$  we have

$$x_\delta \nabla_\delta \phi = r \frac{\partial \phi}{\partial r} = -3\phi.$$

Hence

$$N_\alpha^{(3)} = -\frac{3}{c}\epsilon_{\alpha\beta\gamma}\frac{R}{4\pi}\int d\Omega x_\beta A_\gamma.$$

Comparing this with (A.1) we obtain  $\eta_3 = +1/10$ . Adding together  $\mathbf{N}^{(0)}$  and the surface contributions (7) is obtained.

When the body is situated on the axis of an infinitely long straight ideal solenoid of radius  $R_s$  than (7) remains valid for the field angular momentum within the sphere provided  $R \leq R_s$ . The total field angular momentum, however, is given by (4) (i.e.  $\mathbf{N}^{(0)}$ ) because the surface integrals vanish as  $R \rightarrow \infty$ . Outside the solenoid the vector potential is given by (20) rather than (2) and tends to zero as  $R$  increases. Within the solenoid (2) remains true at arbitrarily large distances from the body but the surface of the parts of the sphere remain finite and the integral vanishes on them for this reason.

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